Journal of Statistical Physics, Vol. 79, Nos. 5/6, 1995

Solving 1D Plasmas and 2D Boundary Problems Using Jack Polynomials and Functional Relations

P. Fendley,[1] F. Lesage,[2] and H. Saleur[1]

Received October 31, 1994

The general one-dimensional "log-sine" gas is defined by restricting the positive and negative charges of a two-dimensional Coulomb gas to live on a circle. Depending on charge constraints, this problem is equivalent to different boundary field theories.

We study the electrically neutral case, which is equivalent to a two-dimensional free boson with an impurity cosine potential. We use two different methods: a perturbative one based on Jack symmetric functions, and a non-perturbative one based on the thermodynamic Bethe ansatz and functional relations. The first method allows us to find an explicit series expression for all coefficients in the virial expansion of the free energy and the experimentally measurable conductance. Some results for correlation functions are also presented. The second method gives an expression for the full free energy, which yields a surprising fluctuation-dissipation relation between the conductance and the free energy.

KEY WORDS: Coulomb gas; exact; conductance; Jack polynomials.

1. INTRODUCTION

The general 2D classical Coulomb gas with charges restricted to live on a circle is a recurrent problem in several areas of theoretical physics. These include random matrix theory,[1] impurity problems (like the Kondo effect[2] and resonant tunneling in quantum wires[3] and between quantum Hall edge states[4,5]), and dissipative quantum mechanics.[6-10] In this 1D "log-sine" gas, the charges interact with a long-range interaction proportional to the log of the sine of the separation.

[1] Department of Physics, University of Southern California, Los Angeles, California 90089-0484.

[2] Service de Physique Théorique de Saclay, F-91191 Gif sur Yvette, France.

0022-4715/95/0600-0799$07.50/0

Two particular cases of this Coulomb gas have been solved analytically. The gas with only one type of charge (the Dyson gas) is related to eigenvalue statistics for circular ensembles[1] and can be addressed by elementary methods. When there are two types of charges required to alternate in space, the gas is related to the Kondo problem[2] and the problem can be addressed indirectly by the Bethe ansatz solution of the Kondo problem[11] (some results are also available from a lattice regularization of the gas[12]).

We present in this paper two methods to address the general model with two types of charges. The first method is direct, and uses the recently developed technical tool of Jack symmetric functions.[13-16] These have been used extensively in recent work on Green functions for the Calogero–Sutherland model.[17-22] The second method is indirect and uses the solution of the boundary sine-Gordon theory[23] via exact S matrices and the thermodynamic Bethe ansatz (TBA).[24,25]

Our results provide two different expressions for the free energy for a large range of couplings, both of which can be evaluated numerically. Previously, only results at a few special points had been available.[26,27]

Technically, the combination of the two methods allows us to compute a rather large number of quantities, including dynamical properties. When they overlap, they can be compared, leading to interesting relations between two different active areas of mathematical physics: $1/r^2$ models and the TBA. Physically, our solution can be applied to a range of interesting problems, including the experimentally measurable resonant tunneling between quantum Hall edge states (which was derived using the TBA in ref. 5) and the case of dissipative quantum mechanics, which will be discussed elsewhere.

We start by describing this model as a $(1+1)$-dimensional field theory with an impurity. Consider a Gaussian model on an infinite cylinder with action

$$A = -\tfrac{1}{2}\int_{-\infty}^{\infty}\int_{0}^{R} d\sigma\, d\tau\, [(\partial_\tau\phi)^2 + (\partial_\sigma\phi)^2] + 2g\int_0^R d\tau \cos[\beta\phi(\sigma=0,\tau)] \tag{1.1}$$

With periodic boundary conditions in the τ direction, this is equivalent to a one-dimensional quantum problem at nonzero temperature $1/R$, with an impurity at $\sigma=0$. The impurity coupling g has a dimension, so this problem is not conformally invariant and the interaction induces a flow from the free boson on an infinite cylinder in the UV ($g=0$) to two decoupled free bosons on two half-cylinders with Dirichlet boundary conditions at their boundary in the IR (g large). A convenient quantity describing this flow is the "g-factor" discussed in ref. 28, whose logarithm is equal to the impurity entropy (the contribution to the entropy which is independent of the length of the cylinder). This "g-factor" (which we prefer to denote ω here) is $\omega = 1$ in the UV and $\omega = t^{-1/2}$ in the IR, where we define

$$t \equiv \frac{4\pi}{\beta^2}$$

We can study (1.1) by naive perturbation theory, which exhibits the relation with a Coulomb gas. Defining as usual the partition function as $Z = \int [d\phi]\, e^{A}$, we introduce

$$\mathscr{Z} \equiv \frac{Z(g)}{Z(g=0)}$$

Using the free field propagator on an infinite cylinder

$$\langle \phi(\tau)\, \phi(\tau') \rangle = -\frac{1}{2\pi} \ln \left| \frac{\kappa R}{\pi} \sin \frac{\pi}{R} (\tau - \tau') \right|$$

(where κ is a renormalization constant), we obtain by the standard perturbation expansion in powers of g

$$\begin{aligned} \mathscr{Z} = \sum_{n=0}^{\infty} \frac{g^{2n}}{(n!)^2} \int_0^R d\tau_1 \cdots d\tau_n\, d\tau'_1 \cdots d\tau'_n \\ \times \prod_{i<j} \left| \frac{\kappa R}{\pi} \sin \frac{\pi(\tau_i - \tau_j)}{R} \right|^{\beta^2/2\pi} \prod_{k<l} \left| \frac{\kappa R}{\pi} \sin \frac{\pi(\tau'_k - \tau'_l)}{R} \right|^{\beta^2/2\pi} \\ \times \left\{ \prod_{i,k} \left| \frac{\kappa R}{\pi} \sin \frac{\pi(\tau_i - \tau'_k)}{R} \right|^{\beta^2/2\pi} \right\}^{-1} \end{aligned} \tag{1.2}$$

The sum (1.2) is the grand canonical partition function for a classical two-dimensional Coulomb gas with *two* species of particles (with positive and negative charges) that lie on a circle of radius R, and which is electrically neutral. The parameter g is the fugacity of charges, while β controls the combination $[\text{charge}]^2 \times [\text{inverse temperature}]$. This expression requires regularization for $\beta^2 \geqslant 4\pi$, but is well defined otherwise. In Section 2 we will show how these integrals for $\beta^2 < 4\pi$ can be explicitly evaluated by expanding the integrands in terms of Jack polynomials.

We can also reformulate this as a boundary problem (i.e., on the half-cylinder) following ref. 29. We introduce two new fields

$$\phi_e(\sigma, \tau) = \frac{1}{\sqrt{2}} [\phi(\sigma, \tau) + \phi(-\sigma, \tau)]$$

$$\phi_o(\sigma, \tau) = \frac{1}{\sqrt{2}} [\phi(\sigma, \tau) - \phi(-\sigma, \tau)]$$

so the action reads now

$$\begin{aligned} A = &-\frac{1}{2} \int_0^\infty \int_0^R d\sigma \, d\tau \, [(\partial_\tau \phi_e)^2 + (\partial_\sigma \phi_e)^2 + (\partial_\tau \phi_o)^2 + (\partial_\sigma \phi_o)^2] \\ &+ 2g \int_0^R d\tau \cos\left[\frac{\beta}{\sqrt{2}} \phi_e(\sigma = 0, \tau)\right] \end{aligned} \tag{1.3}$$

One can of course obtain the perturbative expansion (1.2) from (1.3) by using the propagator on the half-cylinder. In the UV both fields have Neumann boundary conditions and $\omega = (2/t)^{1/2}$, while in the IR the odd field still has Neumann boundary conditions and the even has Dirichlet boundary conditions and $\omega = 1/\sqrt{2}$. Notice that in the reformulation the absolute values of ω have changed; this is presumably due to Jacobian terms in the functional integral when redefining the new fields. However, the ratio $\omega_{\mathrm{UV}}/\omega_{\mathrm{IR}} = \omega_{\mathrm{D}}/\omega_{\mathrm{N}} = t^{-1/2}$ remains a constant.

Using this reformulation, we can obtain the free energy $\mathscr{F} = -T \ln \mathscr{Z}$ nonperturbatively using the thermodynamic Bethe ansatz.[24,25] The corresponding analysis appears in ref. 23 for t integer. We will discuss this in Section 4, where we will also derive a set of functional equations which $\mathscr{Z}$ satisfies. The results of the TBA depend on a dimensionless variable T/T_{B}, where

$$g = \kappa' T_{\mathrm{B}}^{(t-1)/t}$$

and κ' is an unknown renormalization constant. The nonperturbative free energy also contains a nonanalytic term $\bar{\bar{\mathscr{F}}}$ independent of T, which detailed analysis gives as

$$\bar{\bar{\mathscr{F}}} = \frac{T_{\mathrm{B}}}{2 \cos[\pi/2(t-1)]} \tag{1.4}$$

There is also a shift because we have $\mathscr{F}(g=0)=0$, whereas the TBA is defined so that $\mathscr{F}_{\text{TBA}}(g\to\infty)\to 0$. Thus

$$\mathscr{F}_{\text{TBA}} = \mathscr{F} - T\ln\sqrt{t} + \bar{\mathscr{F}} \tag{1.5}$$

This allows us to obtain the exact behavior of the free energy at large g (large T_{B}), because the power series must be precisely balanced by $\bar{\mathscr{F}}$, and thus

$$\mathscr{Z} \approx \frac{1}{\sqrt{t}} \exp\left(\frac{T_{\text{B}}}{2\cos[\pi/2(t-1)]}\right) \tag{1.6}$$

in this limit. We will see in Section 2 that this behavior also follows from the expansion (1.2).

This two-dimensional local field theory problem can also be reformulated as a one-dimensional nonlocal field theory on a circle of circumference R by integrating out the "bulk" degrees of freedom. Let us consider, for instance, the second (boundary) point of view and forget about the odd field, which totally decouples. After integration the even action reads

$$A_e^{\text{bdry}} = -\frac{\pi}{2R^2}\int_0^R \int_0^R d\tau\, d\tau' \frac{\phi_e(\tau)\,\phi_e(\tau')}{[\sin(\pi/R)(\tau-\tau')]^2} + 2g\int_0^R d\tau \cos\left[\frac{\beta}{\sqrt{2}}\phi_e(\tau)\right]$$

We therefore have a one-dimensional model with a sine-Gordon-type interaction, where the Gaussian part has a $1/r^2$ interaction. Related models have been considered in refs. 30 and 31.

2. USING JACK POLYNOMIALS IN THE COULOMB GAS PROBLEM

In this section we find a series expression for all the coefficients of the perturbative expansion of the partition function. The result (1.2) can be easily recast after a change of integration variables into

$$\mathscr{Z} = \sum_{n=0}^{\infty} x^{2n} Z_{2n} \tag{2.1}$$

where we have set

$$x = Rg\left(\frac{\kappa R}{2\pi}\right)^{-1/t}$$

and

$$Z_{2n} \equiv \frac{1}{(n!)^2} \int_0^{2\pi} \frac{du_1}{2\pi} \cdots \frac{du_n}{2\pi} \frac{du'_1}{2\pi} \cdots \frac{du'_n}{2\pi}$$
$$\times \left| \frac{\prod_{i<j} 2\sin((u_i - u_j)/2) \prod_{k<l} 2\sin((u'_k - u'_l)/2)}{\prod_{i,k} 2\sin((u_i - u'_k)/2)} \right|^{2/t}$$
$$= \frac{1}{(n!)^2} \oint \prod_i \left(\frac{dz_i}{2i\pi z_i} \frac{dz'_i}{2i\pi z'_i} \right) \frac{[\Delta(z)\overline{\Delta(z)}]^{1/t} [\Delta(z')\overline{\Delta(z')}]^{1/t}}{\prod_{i,k}[(1 - z_i \bar{z}'_k)(1 - z'_k \bar{z}_i)]^{1/t}} \tag{2.2}$$

where $\Delta(z)$ is the n-variable Vandermonde determinant $\Delta(z) = \prod_{i<j} (z_i - z_j)$, and $z_k = \exp(iu_k)$.

To evaluate this integral, we expand the integrand in terms of Jack polynomials[15,13,14]

$$\prod_{i,j} \frac{1}{(1 - r_i s_j)^a} = \sum_\lambda b_\lambda(a)\, P_\lambda(r, a)\, P_\lambda(s, a) \tag{2.3}$$

The function $P_\lambda(r, a)$ is a symmetric polynomial in the set of variables $(r_1, r_2,..., r_n)$ which depends on a rational number a. The subscript λ is a partition of an integer; this is conveniently labeled by a Young tableau; e.g., the partition $5 = 2 + 2 + 1$ is labeled by the tableau with two boxes in the first row, two boxes in the second, and one in the third. The polynomials $P_\lambda(x, a)$ vanish if the number of parts $l(\lambda)$ of the partition λ (i.e., the number of rows of the tableau) is greater than the number n of variables, so the sum runs over all partitions of all integers with $l(\lambda) \leqslant n$. The Jack polynomials have the useful property that their orthogonality relation involves the Vandermonde determinant:

$$\oint \prod_i \frac{dz_i}{2i\pi z_i} [\Delta(z)\overline{\Delta(z)}]^a P_\lambda(z, a)\, P_\mu(\bar{z}, a) = \delta_{\lambda,\mu} N_\lambda(a) \tag{2.4}$$

Hence the value of the integral follows (here $a = 1/t$):

$$Z_{2n} = \frac{1}{(n!)^2} \sum_{\substack{\lambda \\ l(\lambda) \leqslant n}} b_\lambda^2 N_\lambda^2 \tag{2.5}$$

The numerical coefficients in the foregoing expression are expressed as a product over the boxes of the Young tableau associated with the partition λ.[15,13,14] We have

$$b_\lambda^2 N_\lambda^2 = c_n^2 \prod_{s \in \lambda} \left(\frac{j - 1 + (1/t)(n - i + 1)}{j + (1/t)(n - i)} \right)^2 \tag{2.6}$$

where

$$c_n = \frac{\Gamma(1+n/t)}{[\Gamma(1+1/t)]^n}$$

and $s=(i,j)$ is the box of the tableau at the intersection of the jth column and ith line.

One can write this product compactly using gamma functions. Two convenient expressions of Z_{2n} follow, depending on whether one uses partitions or their conjugates (given by interchanging the rows and columns of the Young tableau). One obtains

$$Z_{2n} = \left(\frac{c_n}{n!}\right)^2 \sum_{\substack{\lambda \\ l(\lambda) \leqslant n}} \prod_{i=1}^{l(\lambda)} \left[\frac{\Gamma((1/t)(n-i)+1)\,\Gamma((1/t)(n+1-i)+\lambda_i)}{\Gamma((1/t)(n-i+1))\,\Gamma((1/t)(n-i)+1+\lambda_i)}\right]^2 \tag{2.7}$$

and alternatively, using the conjugates,

$$\begin{aligned} Z_{2n} = \left(\frac{c_n}{n!}\right)^2 \sum_{\substack{\lambda \\ l(\lambda) \leqslant n}} \prod_{i=1}^{\lambda_1} \left(\frac{\Gamma(n+1+t(i-1))}{\Gamma(n+ti)}\right)^2 \\ \times \left(\frac{\Gamma(n-\lambda'_i+ti)}{\Gamma(n-\lambda'_i+1+t(i-1))}\right)^2 \end{aligned} \tag{2.8}$$

where $\lambda_{i+1} \leqslant \lambda_i$.

Consider, for instance, the case $n=1$: (2.7) reads then

$$Z_2(t) = \sum_{\lambda_1=0}^{\infty} \left(\frac{\Gamma(1/t+\lambda_1)}{\Gamma(1/t)\,\Gamma(1+\lambda_1)}\right)^2 \tag{2.9}$$

The sum converges only for $t>2$, where the UV dimension of the perturbing operator is $x=\beta^2/4\pi<1/2$.[32] This coincides with the domain where the integrals in (1.2) are UV convergent. (We have no problem with IR divergences because we are on a circle.) The sum in (2.9) can be done explicitly; one can also obtain its value by a direct treatment of the integral, avoiding Jack functions. The result is

$$Z_2(t) = \frac{\Gamma(1-2/t)}{\Gamma^2(1-1/t)} \tag{2.10}$$

Let us now study the large-n behavior of the series (1.2). Using the expression with conjugate partitions (2.8), we find

$$Z_{2n} = \left(\frac{c_n}{n!}\right)^2 \sum_{\substack{\lambda \\ l(\lambda) \leqslant n}} \prod_{i=1}^{\lambda_1} \left(\frac{\begin{pmatrix}(n-\lambda'_i+ti-1)(n-\lambda'_i+ti-2)\cdots \\ (n-\lambda'_i+ti-t+1)\end{pmatrix}}{(n+ti-1)(n+ti-2)\cdots(n+ti-t+1)}\right)^2$$

Since conjugate partitions are limited by the number n, we can approximate the sum in (2.8) for large n as an integral over the variables $\lambda'_i/n \equiv v_i$. Calling the number of boxes in the first line $\lambda_1 \equiv p$, we get

$$Z_{2n} \simeq \left(\frac{c_n}{n!}\right)^2 \sum_{p=0}^{\infty} N^p \int_0^1 dv_1 \int_0^{v_1} dv_2 \cdots \int_0^{v_{p-1}} dv_p$$
$$\times \prod_{i=1}^{p} \left(1 - \frac{v_i}{(1+ti/N)}\right)^{2t-2}$$

The integrand is minimized by $(1-v_i)^{2t-2}$ and maximized by 1. In both cases one can symmetrize over the v_i to compute the integral explicitly and one finds

$$e^{n/(2t-1)} \left(\frac{c_n}{n!}\right)^2 < Z_{2n} < e^n \left(\frac{c_n}{n!}\right)^2$$

Therefore, the large-n behavior of the Z_{2n} is fully controlled by the $(c_n/n!)^2$ prefactor and

$$Z_{2n} \approx \exp\left[2n\left(\frac{1}{t}-1\right)\log n + O(n)\right] \tag{2.11}$$

This result was previously derived in ref. 33. An important conclusion is that for $t>2$ the radius of convergence of (2.1) is *infinite*. Moreover, approximating the sum over n by an integral, we find

$$\mathscr{Z} \approx \exp(\text{const} \cdot x^{t/(t-1)}) \tag{2.12}$$

which is in agreement with (1.6). This behavior is well expected on physical grounds. Indeed the partition function reads also $\mathscr{Z} = \exp(\mathscr{E}/T - \mathscr{S})$, where $T = 1/R$ is the temperature of the equivalent one-dimensional quantum system, and $\mathscr{E}$ and $\mathscr{S}$ are the impurity energy and entropy, respectively. In the deep IR the impurity entropy converges to $s \to \ln \omega$ and the energy, on dimensional grounds, scales as $g^{t/(t-1)}$. The behavior (2.12) is the analog of the "bulk term" in flows between bulk critical points.

Although expression (2.7) or (2.8) is in effect a solution of the problem, one can wonder about its practical use. Trying to evaluate the Z_{2n} numerically, one finds that the series converges very slowly. For example, for $t=3$, evaluating a billion terms gives an accuracy of only about 0.1%.

Fortunately, results are greatly improved by studying the free energy $\mathscr{F} = -T \ln \mathscr{Z}$, whose expansion we write

$$\mathscr{F} = T \sum_{n=0}^{\infty} f_{2n} x^{2n} \tag{2.13}$$

The f_{2n} are of course given in terms of the Z_{2n}. For example, $f_2 = -Z_2$ and $f_4 = -Z_4 + Z_2^2/2$. When evaluating the f_{2n} numerically for $n > 1$, we find a much faster convergence. For $t = 3$, we have

$$\begin{aligned} f_2 &= -\Gamma(1/3)/\Gamma^2(2/3), & f_8 &= 0.44223558 \\ f_4 &= 0.229454064, & f_{10} &= -0.022852208 \\ f_6 &= -0.092261103, & f_{12} &= 0.012329254 \end{aligned} \tag{2.14}$$

The Z_{2n} can be extracted from these data, and are

$$\begin{aligned} Z_2 &= \Gamma(1/3)/\Gamma^2(2/3), & Z_8 &= 0.0618476490 \\ Z_4 &= 0.8378042270, & Z_{10} &= 0.01021005440 \\ Z_6 &= 0.276783312, & Z_{12} &= 0.00131673987 \end{aligned} \tag{2.15}$$

These coefficients are enough to get a good approximation of the properties all the way to the infrared (very large x) using Padé approximants. It is then preferable to consider the entropy $\mathscr{S} = \partial\mathscr{F}/\partial T$, which is bounded for x large. Keeping the coefficients through f_{12}, one finds, for instance, that $\mathscr{S}(x=0) - \mathscr{S}(x=\infty) \approx 0.57$, in good agreement with the exact value $\ln \sqrt{3} \approx 0.549306\ldots$.

3. OTHER QUANTITIES OF INTEREST

The previous calculation of the partition function is the simplest calculation which can be done using Jack symmetric functions. In this section we present several other calculations, and present a conjecture for the experimentally measurable conductance.

3.1. Twisted Partition Functions

We have so far considered a periodic field ϕ on the cylinder. We could also have winding modes such that

$$\phi(\sigma, \tau + R) = \phi(\sigma, \tau) + \frac{2\pi}{\beta} p \tag{3.1}$$

where p is an integer. By splitting the field into classical and quantum parts we obtain an action similar to (1.1), but the interaction term is now

$$2g \int_0^R d\tau \cos\left[\beta\phi(\sigma=0,\tau)+2\pi\frac{p}{R}\tau\right]$$

Defining as before $\mathscr{Z}(p)\equiv Z(g,p)/Z(g=0,p)$, we find a perturbative expansion similar to (1.2) with, however, each term in the sum multiplied by

$$\exp\left[i2\pi\frac{p}{R}(\tau_1+\cdots+\tau_n-\tau'_1\cdots-\tau'_n)\right]$$

After change of variables we have the same expansion as (2.1), but with

$$\begin{aligned} Z_{2n}(p)\equiv{}&\frac{1}{(n!)^2}\oint\prod_i\left(\frac{dz_i}{2i\pi z_i}\frac{dz'_i}{2i\pi z'_i}\right)\\ &\times\frac{[\Delta(z)\overline{\Delta(z)}]^{1/t}[\Delta(z')\overline{\Delta(z')}]^{1/t}}{\prod_{i,k}[(1-z_i\bar{z}'_k)(1-z'_k\bar{z}_i)]^{1/t}}\left(\frac{z_1\cdots z_n}{z'_1\cdots z'_n}\right)^p \end{aligned}\tag{3.2}$$

so in the Coulomb-gas language there is now a magnetic charge located at the center of the circle. (We assume p is positive, otherwise just replace p by $|p|$.) We now use the fact that

$$(z_1\cdots z_n)^p P_\lambda(z,a)=P_{\lambda+p}(z,a)\tag{3.3}$$

where $\lambda+p$ means the partition λ where p columns of length n have been added. Therefore

$$Z_{2n}(p)=\frac{1}{(n!)^2}\sum_{\substack{\lambda\\ l(\lambda)\leqslant n}} b_\lambda b_{\lambda+p}N_\lambda^2\tag{3.4}$$

where we use the relation $N_\lambda=N_{\lambda+p}$ that follows immediately from the integral (2.4) defining the norm N. The coefficients b_λ read

$$b_\lambda=\prod_{s\in\lambda}\frac{\lambda_i-j+(1/t)(\lambda'_j-i+1)}{\lambda_i-j+1+(1/t)(\lambda'_j-i)}$$

For instance, one has

$$\begin{aligned} Z_2(\alpha)&=\sum_{\lambda_1=0}^{\infty}\frac{\Gamma(1/t+\lambda_1)\,\Gamma(1/t+\lambda_1+\alpha)}{\Gamma^2(1/t)\,\Gamma(1+\lambda_1)\,\Gamma(1+\lambda_1+\alpha)}\\ &=\frac{\sin(\pi/t)\,\Gamma(1-2/t)}{\sin\pi(1/t+\alpha)\,\Gamma(1-1/t+\alpha)\,\Gamma(1-1/t-\alpha)} \end{aligned}\tag{3.5}$$

for α an arbitrary real number. When $\alpha = p$ is an integer,

$$Z_2(p) = \frac{(-1)^p \Gamma(1-2/t)}{\Gamma(1-1/t+p)\,\Gamma(1-1/t-p)} \tag{3.6}$$

We can also compute the partition function when electrical neutrality is broken by some amount Q (assumed integer). Defining

$$\mathscr{Z}_{\mathscr{Q}} = \lim_{\sigma \to \infty} \sigma^{Q^{\beta^2/2\pi}} \frac{\int [d\phi]\, e^{iQ\phi(\sigma,0)} e^{A}}{Z(g=0)}$$

we have

$$\mathscr{Z}_{\mathscr{Q}} = \sum_{n=0}^{\infty} x^{2n+Q} Z_{n,n+Q} \tag{3.7}$$

where $Z_{n,n+Q}$ has formally the same expression as (2.2), but there are n variables z and $n+Q$ variables z'. By the same manipulations we obtain

$$Z_{n,n+Q} = \frac{1}{n!\,(n+Q)!} \sum_{\substack{\lambda \\ l(\lambda) \leqslant n}} b_\lambda(n)\, b_\lambda(n+Q)\, N_\lambda(n)\, N_\lambda(n+Q) \tag{3.8}$$

Observe that for $n = 0$ we recover the well-known expression[1]

$$Z_{0,Q} = b_0(Q)\, N_0(Q) = c_Q$$

3.2. Correlation Functions and the Conductance

The tool of Jack polynomials should allow the perturbative evaluation of correlation functions. However, the calculation requires the knowledge of branching coefficients which are not known yet in general. We will illustrate this with an example. Consider the two-point function of the field ϕ itself. By the same perturbative approach we find

$$\langle \phi(\tau)\phi(\tau') \rangle_g = \langle \phi(\tau)\phi(\tau') \rangle_0 + \frac{1}{\mathscr{Z}} \frac{1}{4\pi t} \sum_{n=1}^{\infty} \frac{x^{2n}}{(n!)^2} \sum_{p=1}^{\infty} \frac{(\bar{z}z')^p + (z\bar{z}')^p}{p^2} \mathscr{C}_{np} \tag{3.9}$$

where

$$\mathscr{C}_{np} \equiv \oint \prod_{i=1}^{n} \left(\frac{dz_i}{2i\pi z_i} \frac{dz'_i}{2i\pi z'_i} \right) \frac{(\Delta(z)\,\overline{\Delta(z)})^{1/t}\, (\Delta(z')\,\overline{\Delta(z')})^{1/t}}{\prod_{i,k} [(1-z_i\bar{z}'_k)(1-z'_k\bar{z}_i)]^{1/t}}$$
$$\times R_p(z_i, z'_i)\, R_p(\bar{z}'_i, \bar{z}_i)$$

$$R_p(x_i, y'_i) \equiv \sum_i (x_i^p - y_i^p)$$

The calculation can be easily done for the first term $n=1$ using (3.3). One finds

$$\mathscr{C}_{1_p}=2[Z_2(p)-Z_2(0)] \tag{3.10}$$

For general n, we can still decompose the Coulomb-interaction term between positive and negative charges using Jack polynomials as in (2.3). However, we cannot use the relation (2.4) since we have the extra factors R_p. We can decompose[16,22]

$$\sum_i z_i^p=\frac{p}{a}\sum_{|\lambda|=p}\frac{\prod_{(i,j)\neq(1,1)}[(j-1)/a-(i-1)]}{\prod_{(i,j)}[\lambda'_j-i+(\lambda_i-j+1)/a]}P_\lambda(z,a) \tag{3.11}$$

where $|\lambda|=\sum_i\lambda_i$ (the total number of boxes in the tableau). As before, $a=\beta^2/4\pi=1/t$ and the only tableaux which contribute are the ones of the form $\lambda_1\geqslant\lambda_2\cdots\geqslant\lambda_n$. The Jack polynomials in (3.11) multiply the ones in (2.3). We therefore end up with the problem of determining the coupling coefficients

$$P_\lambda P_\mu=\sum_\nu g^\nu_{\lambda\mu}P_\nu$$

Unfortunately these coefficients are not yet known in general.

The two-point function of the field ϕ is especially useful in physical applications: in the 2D boundary problem the Kubo formula relates it to the conductance,[3] while in dissipative quantum mechanics it is the mobility.[8,9] The conductance at the Matsubara frequency $\omega_p=(2\pi/R)p$ follows from

$$G_n=\frac{2\omega_p}{t}\int_0^R d\tau'\,\langle\phi(\tau)\,\phi(\tau')\rangle\,e^{(2i\pi/R)p(\tau-\tau')} \tag{3.12}$$

The g^2 term is easily picked up using (3.10). The dc conductance then is obtained by analytically continuing (3.10) to $p=0$, leading to

$$G=\frac{1}{t}+2\frac{x^2}{t^2}\operatorname*{Lim}_{n\to 0}\frac{Z_2(n)-Z_2(0)}{n}+O(x^4)$$

Using (3.5), it is easy to perform the limit and one finds finally

$$G=\frac{1}{t}-\frac{x^2}{t^2}2^{1-2/t}\frac{\Gamma(1/t)\,\Gamma(1/2)}{\Gamma(1/2+1/t)}+O(x^4) \tag{3.13}$$

in agreement with the integral done without the Jack functions.[3]

We cannot for the moment compute the two-point function to all orders. However, we have the following conjecture for G to all orders:

$$G = \frac{1}{t} + \frac{x}{t^2} \frac{d^2}{d\alpha\, dx} \ln Z(g, \alpha) \bigg|_{\alpha=0} \tag{3.14}$$

so the first few terms read then

$$G = \frac{1}{t} + \frac{2}{t^2} [Z_2' x^2 + 2(Z_4' - Z_2 Z_2')\, x^4$$
$$+ 3(Z_6' - Z_2' Z_4 - Z_4 Z_2' + Z_1' Z_2^2)\, x^6 + \cdots]$$

We define the continuation of $Z_{2n}(p)$ to real values of p by simple substitution in (3.4), so

$$Z_{2n}' \equiv \frac{d}{d\alpha} Z_{2n}(\alpha) \bigg|_{\alpha=0}$$
$$= \frac{1}{(n!)^2} \sum_{\substack{\lambda \\ l(\lambda) \leqslant n}} b_\lambda^2 N_\lambda^2 \left[\sum_{i=1}^{n} \psi\left(\lambda_i + \frac{1}{t}(n-i+1)\right)\right.$$
$$\left. - \psi\left(\lambda_i + 1 + \frac{1}{t}(n-i)\right)\right] \tag{3.15}$$

where $\psi(z) = \Gamma(z)'/\Gamma(z)$. We can investigate these numbers numerically. For $t = 3$, (3.15) gives $Z_2' = -2.64996$, $Z_4' = -2.351$, and $Z_6' = -0.964$. The first agrees with (3.13), and the others are in very good agreement with the conductance calculated using the TBA in the next section. The conjecture (3.14) appears to be a reasonable form of the Kubo formula,[3] but we have not succeeded in deriving it rigorously.

To close this section, we remark that, although the theory of Jack symmetric functions generally deals with rational values of t, we expect all formulas obtained above to hold for any $t > 2$ by naive substitution.

4. NONPERTURBATIVE TREATMENT

The boundary problem (1.3) is integrable.[34] One can thus find the exact S matrix for the quasiparticles of the problem[34] and then use the thermodynamic Bethe ansatz[24,25] to compute the free energy[23] and the conductance.[5] In this section, we describe these results and use them to derive a variety of functional equations. These functional relations give nonperturbative equations for the free energy and allow one to derive simple but nontrivial relations among the coefficients Z_{2n}. Another functional relation relates the conductance to the free energy, thus giving a new fluctuation-dissipation theorem for this system. We also find expansion coefficients at $t=3$ numerically from the TBA as another check on our results.

The starting point of the TBA is the quasiparticle description of a two-dimensional integrable field theory. These quasiparticles scatter among themselves and off of the impurity with a known S matrix. At any value of t, the quasiparticle spectrum includes the soliton and antisoliton, which we label by $+$ and $-$, respectively. Moreover, at coupling t, there are $t-2$ "breather" states in the spectrum. The energy and momentum of these left-moving massless particles are parametrized by rapidity variable θ, so $E=-P=\mu_r \exp(-\theta)$, where $\mu_+ = \mu_- = \mu$ and $\mu_j = 2\mu \sin[\pi/2(t-1)]$ for the breathers. We define the density of states n_r and the density of filled states ρ_r for each quasiparticle species r. Periodic boundary conditions give the n_r as a functional of the ρ_r. The free energy can be written in terms of these quantities; demanding it be at a minimum gives another set of relations which determine the densities. These relations are most conveniently written in terms of the functions $\varepsilon_r(\theta)$, which are defined by

$$\frac{1}{1+e^{\varepsilon_r}} \equiv \frac{\rho_r}{n_r}$$

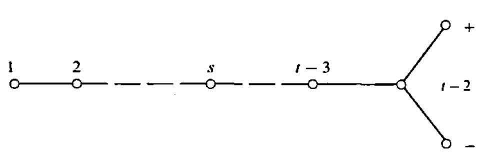
Notice that if the particles are free, ρ_r/n_r is the Fermi distribution function. However, for $t \neq 2$, the particles are not free, and the ε_r are determined by the TBA equation

$$\varepsilon_r(\theta) = \frac{\mu_r}{\mu} e^{-\theta} - \frac{1}{2\pi} \sum_s \int_{-\infty}^{\infty} d\theta' \, \varphi_{rs}(\theta-\theta') \ln(1+e^{-\varepsilon_s(\theta')}) \tag{4.1}$$

where the label s runs over breathers $(1 \cdots t-2)$ and $\pm$. The functions φ_{rs} for integer t are given in refs. 35 and 23. We will not need them, because here these equations can be written in a much simpler form[36]:

$$\varepsilon_r = \int_{-\infty}^{\infty} d\theta' \frac{(t-1)}{2\pi \cosh[(t-1)(\theta-\theta')]} \sum_s N_{rs} \ln(1+e^{\varepsilon_s(\theta')}) \tag{4.2}$$

where N_{rs} is the incidence matrix of the following diagram:



The dependences on the ratios μ_r/μ seem to have disappeared from (4.2), but they appear as an asymptotic condition: the original equations (4.1) indicate that the solution must satisfy

$$\varepsilon_r \to \frac{\mu_r}{\mu} e^{-\theta} \qquad \text{as} \quad \theta \to -\infty \tag{4.3}$$

4.1. The Partition Function

The impurity free energy is given in terms of ε_+:

$$\mathscr{F}_{\mathrm{TBA}} = \frac{T_{\mathrm{B}}}{2\cos[\pi/2(t-1)]} - T\int \frac{d\theta}{2\pi} \frac{t-1}{\cosh[(t-1)(\theta-\alpha)]} \ln(1+e^{\varepsilon_+(\theta)}) \tag{4.4}$$

where $\alpha \equiv \ln(T/T_{\mathrm{B}})$.[3] The first piece is the nonanalytic term (1.4). Since the same kernel appears in (4.4) and (4.2), $\mathscr{F}$ and $\mathscr{Z}$ can be written in a simpler form for many of the t. Using the relation (1.5) between $\mathscr{F}$ and $\mathscr{F}_{\mathrm{TBA}}$, we have, for example,

$$\begin{aligned} \mathscr{Z}(\alpha) &= [Y_1(\alpha)/3]^{1/2}, && t=3 \\ \mathscr{Z}(\alpha) &= \tfrac{1}{2}[Y_2(\alpha)]^{1/3}, && t=4 \\ \mathscr{Z}(\alpha) &= [Y_3(\alpha)/5Y_1(\alpha)]^{1/2}, && t=5 \end{aligned} \tag{4.5}$$

where we define $Y_r \equiv \exp(\varepsilon_r)$.

We derive simple functional relations for $\mathscr{Z}(\alpha)$ by continuing it and the $Y_r(\alpha)$ into the complex α plane.[36] Using the simple identity

$$\lim_{x\to 0}\left[\frac{\lambda}{\cosh(\lambda\theta + i\pi/2 - x)} + \frac{\lambda}{\cosh(\lambda\theta - i\pi/2 + x)}\right] = 2\pi\delta(\theta) \tag{4.6}$$

we have

$$\mathscr{Z}(\alpha+\gamma)\,\mathscr{Z}(\alpha-\gamma) = \frac{1}{t}[1+Y_+(\alpha)] \tag{4.7}$$

where $\gamma \equiv i\pi/2(t-1)$. Similarly, Eq. (4.2) yields

$$\begin{aligned} Y_+(\theta+\gamma)\, Y_+(\theta-\gamma) &= 1 + Y_{t-2}(\theta) \\ Y_{t-2}(\theta+\gamma)\, Y_{t-2}(\theta-\gamma) &= [1+Y_{t-3}(\theta)][1+Y_+(\theta)]^2 \\ Y_a(\theta+\gamma)\, Y_a(\theta-\gamma) &= [1+Y_{a+1}(\theta)][1+Y_{a-1}(\theta)] \end{aligned} \tag{4.8}$$

[3] This can be derived from the kernels κ_a of ref. 23 by using (4.2) along with the identity $2\cosh y\tilde{\kappa}_a = \sum_b N_{ab}\tilde{\kappa}_b$ for $a = 1 \cdots t-2$ and $2\cosh y(\tilde{\kappa}_+ + \kappa_-) = 2\tilde{\kappa}_{t-2} + 1$.

where $a = 1 \cdots t-3$, we define $Y_0 \equiv 0$, and it follows from symmetry that $Y_+ = Y_-$. These equations are applicable everywhere in the complex θ plane, whereas the original TBA equations apply only in a strip $|\mathrm{Im}\,\theta| < \pi/(t-1)$. The functional relations determine the functions Y_r and $\mathscr{X}$ once the asymptotic condition (4.3) is imposed. One can argue[36] that the functions $Y_r(\alpha)$ [and $\mathscr{X}(\alpha)$] have the periodicity $Y_r(\alpha + t\gamma) = Y_r(\alpha)$, which implies that they can be expanded in powers of $(T_B/T)^{-2(t-1)/t}$ for T_B/T small. This then gives the expansions (2.1) and (2.13), because $x \propto (T_B/T)^{(t-1)/t}$.

Plugging the perturbative expansions into the functional relations (4.7) and (4.8) gives nontrivial relations among the coefficients determined in Section 2 by the Jack-polynomial expansions. For $t = 3, 4$ these constraints can be written in a simple form by using (4.5). Doing a little algebra, we have

$$
\begin{aligned}
&3\mathscr{X}(\alpha + i\pi/2)\, \mathscr{X}(\alpha - i\pi/2)\, \mathscr{X}(\alpha) \\
&\qquad = \mathscr{X}(\alpha + i\pi/2) + \mathscr{X}(\alpha) + \mathscr{X}(\alpha - i\pi/2), \qquad t = 3 \\
&4\mathscr{X}(\alpha + i\pi/3)\, \mathscr{X}(\alpha - i\pi/3)\, \mathscr{X}(\alpha) \\
&\qquad = \mathscr{X}(\alpha + i\pi/3) + 2\mathscr{X}^2(\alpha) + \mathscr{X}(\alpha - i\pi/3), \qquad t = 4
\end{aligned}
\tag{4.9}
$$

These relations have the nice feature that they have lost all trace of the quasiparticle index r. This is a strong hint that they can be derived directly, without having to do the full TBA analysis. One can also hope that there is a simple relation even for noninteger t (where the TBA analysis can get quite complicated); we make a conjecture in the next subsection. Even though (4.9) are stronger relations than (4.7) and (4.8), this still probably is not the end of the story: the conjecture for $t = 4$ amounts to $2\mathscr{X}(\alpha)\, \mathscr{X}(\alpha + 2i\pi/3) = \mathscr{X}(\alpha + i\pi/3) + \mathscr{X}(\alpha - i\pi/3)$, which yields (4.9), but not the other way around.

Plugging (2.1) into (4.9), one finds

$$
\exp\left(-3 \sum_n f_{6n} x^{6n}\right) = \sum_n Z_{6n} x^{6n}, \qquad t = 3
$$

This means, for example, that for $t = 3$, $2Z_6 = -6f_6 = 3Z_2 Z_4 - (Z_2)^3$ and $Z_{12} = -3f_{12} + 9(f_6)^2/2$. Both agree with the Jack-polynomial expressions numerically evaluated in (2.14) and (2.15). In general, it means that for $t = 3$, the coefficients Z_{6n} are given in terms of the lower coefficients. Similarly, for $t = 4$ one finds that the coefficients Z_{4n} are determined in terms of lower coefficients. For example, $Z_4 = (Z_2)^2/3 = \pi/[3\Gamma^4(3/4)]$ and $9Z_8 = 18Z_2 Z_6 - Z_2^4$. It would certainly be interesting to have a direct proof [i.e., one depending only on the expression (2.7) or (2.8)] of these relations. They are certainly a hint of a much deeper structure to the problem.

Since we cannot determine all of the coefficients from the functional relations, a final check is to solve the TBA equations numerically and then fit the results to a power series. Doing a perturbative expansion of the nonperturbative solution, one obtains

$$\mathscr{F}_{\mathrm{TBA}} = T \sum_{n=0}^{\infty} k_{2n} \left(\frac{T_{\mathrm{B}}}{T}\right)^{(1-1/t)2n} + \bar{\mathscr{F}} \tag{4.10}$$

We can now match these results with those of the Jack-polynomial expansion, once comparison of the first order has determined the ratio of the unknown constants κ, κ'. We expect

$$\frac{k_{2n}}{f_{2n}} = \left(\frac{k_2}{f_2}\right)^n \tag{4.11}$$

We evaluate the full function $\mathscr{F}(T)$ to double-precision accuracy by solving the integral equation (4.1), plugging this into the free energy (4.4), and then fitting this to the series (4.11) at large T. For $t=3$, we find

$$\begin{aligned} &k_2 = -0.4567084, &&k_8 = 0.000422 \\ &k_4 = 0.0224220, &&k_{10} = -0.00007 \\ &k_6 = -0.002818, \end{aligned} \tag{4.12}$$

To the accuracy of the TBA fit, we have excellent agreement. The scales x and T_{B} are therefore related for $t=3$ by

$$x^2 = |k_2| \frac{\Gamma^2(2/3)}{\Gamma(1/3)} \left(\frac{T_{\mathrm{B}}}{T}\right)^{4/3} \tag{4.13}$$

4.2. The Conductance

In this subsection we use the nonperturbative TBA to derive a remarkable fluctuation-dissipation relation of the conductance to the partition function. This allows us to obtain the value of infinitely many coefficients in the perturbation expansion of G. It also allows us to conjecture a functional relation for $\mathscr{Z}$ for any rational value of t.

The TBA gives the conductance as[5]

$$G(\alpha) = \int_{-\infty}^{\infty} d\theta \frac{t-1}{2\cosh^2[(t-1)(\theta-\alpha)]} \frac{1}{1+Y_+(\theta)} \tag{4.14}$$

where $\alpha = \ln(T/T_B)$ as before, and Y_+ is given by the TBA equation (4.2). By using the relation

$$\lim_{x \to 0} \left[\frac{\lambda^2}{\cosh^2(\lambda\theta + i\pi/2 - x)} - \frac{\lambda^2}{\cosh^2(\lambda\theta - i\pi/2 + x)} \right] = -i2\pi\delta'(\theta)$$

one finds

$$G(\alpha+\gamma) - G(\alpha-\gamma) = -i \frac{\pi}{t-1} \frac{\partial}{\partial\alpha} \frac{1}{1+Y_+(\alpha)}$$

where $\gamma \equiv i\pi/[2(t-1)]$. Using the relation (4.7), this gives

$$G(\alpha+\gamma) - G(\alpha-\gamma) = -i \frac{\pi}{t(t-1)} \frac{\partial}{\partial\alpha} [\mathscr{Z}^{-1}(\alpha+\gamma)\, \mathscr{Z}^{-1}(\alpha-\gamma)] \qquad (4.15)$$

This fluctuation-dissipation relation has lost all trace of the quasiparticles of the TBA: it is thus tempting to conjecture that it holds for all t, not just the integer values where the TBA analysis is valid.

The perturbative expansion is $G_{\text{pert}} = \sum_n g_{2n} x^{2n}$, so we have, for example,

$$g_2 = -Z_2 \frac{2\pi}{t^2} \cot \frac{\pi}{t}$$

in agreement with the perturbative calculations (2.10) and (3.13), which are valid for any $t > 2$. When $t \equiv p/q$ is rational, the fluctuation-dissipation relation (4.15) gives many but not all of the g_{2n} in terms of the Z_{2m} (with $m \leqslant n$), because the terms on the left-hand side vanish when n is a multiple of p. For the physically important value $t = 3$, this gives all coefficients g_{6n+2} and g_{6n+4}; for example,

$$g_2 = -\frac{2\pi}{9\sqrt{3}} Z_2, \qquad g_4 = \frac{4\pi}{9\sqrt{3}} Z_4, \qquad g_8 = \frac{8\pi}{9\sqrt{3}} (Z_8 - Z_2 Z_6)$$

These are in excellent agreement with a numerical calculation of the TBA conductance.

Although the fact that some of the terms on the left-hand side of (4.15) vanish for rational t means we do not know how to relate these g_{2n} to the Z_{2n}, it does seem to imply a constraint on Z. We know that Z is an analytic function of x for $t > 2$, but we do not know that G is as well. [One way of checking this would be to check that the explicit perturbative expansions (3.15) and (2.7) for G and $\mathscr{Z}$ obey the formula (4.15).] If G is indeed analytic (so $G = G_{\mathrm{pert}}$), then it requires that these terms on the right-hand side also vanish, which means that $\mathscr{X}$ should satisfy

$$\sum_{j=1}^{p} \mathscr{X}^{-1}(\alpha + 2j\gamma)\, \mathscr{X}^{-1}(\alpha + 2(j-1)\,\gamma) = p \tag{4.16}$$

For $t = 3$ this is the relation already derived in (4.9), but for $t = 4$ it is different. Putting it together with (4.9) for $t = 4$, we find the simpler relation $2\mathscr{X}(\alpha)\, \mathscr{X}(\alpha + 2i\pi/3) = \mathscr{X}(\alpha + i\pi/3) + \mathscr{X}(\alpha - i\pi/3)$. This relation alone implies both (4.16) and (4.9) for $t = 4$. We can check the relationship (4.16) numerically. Plugging the expansion of $\mathscr{X}$ into (4.16) gives the coefficients Z_{2p} in terms of lower ones, which can be compared with the expression (2.7). We have checked Z_{10} for $t = 5$ and $t = 5/2$, and find that it is indeed satisfied. This leads us to conjecture that (4.16) is true for all $t > 2$ and rational. We also note that the relations (4.7) and (4.8) require that $\mathscr{X}$ should obey an even more restrictive functional relationship. We have not succeeded in finding its general form, but one can always plug the perturbative expansions into (4.7) and (4.8) to derive more relations among the coefficients.

To conclude this section, we recall first that the TBA analysis is usually made for t rational only. Moreover, some of the results given above hold for t integer only. However, some of the functional relations we have uncovered seem to make sense for any t. Observe also that the Jack expansion and TBA behave differently as $t \to 2$. In the former case all integrals just blow up, while in the latter case one gets finite results for the free energy, involving, however, logarithmic terms. Presumably, the TBA gives a regularized version of the Jack computations. It remains to be seen if this is equivalent to the lattice regularization at $t = 2$ in ref. 27.

5. CONCLUSION

Using Jack polynomials and the thermodynamic Bethe ansatz, many properties of the 1D log-sine gas can be computed exactly, some of which are of experimental significance. We hope that these methods can be used for other problems with potential applications, in particular, for dissipative quantum mechanics. Moreover, multiple integrals similar to those we do using the Jack symmetric functions appear in many different kinds of computations, so we hope that these techniques are generally applicable.

On the more formal side, it is exciting to have an example where two different areas of mathematical physics meet. By analogy, one might hope that these TBA techniques can be applied to other $1/r^2$ models, such as the Calogero–Sutherland model, where Jack polynomials have been used recently. This overlap of techniques has led to intriguing relations between various quantities of Jack symmetric function theory. For example, when $t=4$ the series (2.7) can be summed to give $Z_4 = \pi/[3\Gamma^4(3/4)]$, but we have no direct proof, so we do not know if this is a fluke or if a closed-form expression can be found for all t. The overlap has also led to several simple but powerful conjectures like (4.16) and (3.14). One can hope that this is evidence of a more complete mathematical structure behind the scene.

ACKNOWLEDGMENTS

H.S. is grateful to Saclay for its warm hospitality when this work was started, and to I. Kostov and V. Pasquier for very useful discussions. F.L. thanks P. Di Francesco for discussions. This work was supported by the Packard Foundation, the National Young Investigator program (NSF-PHY-9357207), the DOE (DE-FG03-84ER40168), and a Canadian NSERC 67 scholarship (F.L.). H.S. is a Packard Fellow.

REFERENCES

1. M. L. Mehta, *Random matrices* (Academic Press, New York, 1991).
2. P. W. Anderson, G. Yuval, and D. R. Hamman, *Phys. Rev. B* **1**:4464 (1970); K.-D. Schotte, *Z. Physik* **235**:155 (1970).
3. C. L. Kane and M. P. A. Fisher, *Phys. Rev. B* **46**:15233 (1992).
4. K. Moon, H. Yi, C. L. Kane, S. M. Girvin, and M. P. A. Fisher, *Phys. Rev. Lett.* **71**:4381 (1993).
5. P. Fendley, A. Ludwig, and H. Saleur, Exact conductance through point contacts in the $\nu = 1/3$ fractional quantum Hall effect, *Phys. Rev. Lett.*, to appear; cond-mat/9408068.
6. A. Schmid, *Phys. Rev. Lett.* **51**:1506 (1983).
7. M. P. A. Fisher and W. Zwerger, *Phys. Rev. B* **32**:6190 (1985).
8. F. Guinea, V. Hakim, and A. Muramatsu, *Phys. Rev. Lett.* **54**:263 (1985).
9. F. Guinea, *Phys. Rev. B* **32**:7518 (1985).
10. C. Callan and D. Freed, *Nucl. Phys. B* **374**:543 (1992).
11. N. Andrei, K. Furuya, and J. Lowenstein, *Rev. Mod. Phys.* **55**:331 (1983); A. M. Tsvelick and P. B. Wiegmann, *Adv. Phys.* **32**:453 (1983).
12. P. J. Forrester, *J. Stat. Phys.* **45**:153 (1986).
13. I. G. Macdonald, *Seminaire Lotharingien* (Publ. IRMA, Strasbourg, 1988).
14. I. G. Macdonald, *Symmetric Functions and Hall Polynomials* (Clarendon Press, Oxford, 1979).
15. R. P. Stanley, *Adv. Math.* **77**:76 (1989).
16. K. W. J. Kadell, *Compos. Math.* **87**:5 (1993).
17. F. Calogero, *J. Math. Phys.* **10**:2191 (1969).
18. B. Sutherland, *J. Math. Phys.* **12**:246, 251 (1977).
19. F. D. M. Haldane, *Phys. Rev. Lett.* **66**:1529 (1991).
20. P. J. Forrester, *Nucl. Phys. B* **388**:671 (1992).
21. F. Lesage, V. Pasquier, and D. Serban, *Nucl. Phys. B* **435**:585 (1995); hep-th/9405008.
22. Z. N. C. Ha, *Phys. Rev. Lett.* **73**:1574 (1994); cond-mat/9405063.

23. P. Fendley, H. Saleur, and N. P. Warner, *Nucl. Phys. B* **430**:577 (1994); hep-th/9406125.
24. C. N. Yang and C. P. Yang, *J. Math. Phys.* **10**:1115 (1969).
25. Al. B. Zamolodchikov, *Nucl. Phys. B* **342**:695 (1991).
26. M. Gaudin, *J. Phys.* (Paris) **46**:1027 (1985).
27. P. J. Forrester, *J. Stat. Phys.* **51**:457 (1988); **54**:57 (1989); **60**:203 (1990).
28. I. Affleck and A. Ludwig, *Phys. Rev. Lett.* **67**:161 (1991).
29. E. Wong and I. Affleck, *Nucl. Phys. B* **417**:403 (1994); cond-mat/9311040.
30. J. L. Cardy, *J. Phys. A* **14**:1407 (1981).
31. K. H. Kjaer and H. J. Hilhorst, *J. Stat. Phys.* **28**:621 (1982).
32. C. Itzykson and H. Saleur, *J. Stat. Phys.* **48**:449 (1987).
33. H. Schultz, *J. Phys. A* **14**:3277 (1981).
34. S. Ghoshal and A. B. Zamolodchikov, *Int. J. Mod. Phys. A* **9**:3841 (1994); hep-th/9306002.
35. T. Klassen and E. Melzer, *Nucl. Phys. B* **338**:485 (1990).
36. Al. B. Zamolodchikov, *Phys. Lett. B* **253**:391 (1991).